An Approach for Extending Dynamic Structural Models to Multi-Product Firms

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Abstract

We propose an approach to extend the standard framework of dynamic games to deal with multi-product firms. Our approach applies to industries with a large number of products offered by a small number of firms.

1 Introduction

Ericson and Pakes (1995) propose what has since become the standard framework for dynamic games. In principle, the parameters of their model, such as investment or sunk costs, can be estimated by maximizing the likelihood of observed choices, following the nested algorithm (Rust, 1987), which has been used successfully in single agent models. In practice, however, this approach is not computationally feasible when studying dynamic games because of the need to solve the equilibrium many times. More recently several alternatives have emerged. A common feature of these new methods is that they avoid the use of computationally-intense techniques to compute the equilibrium strategies, and instead estimate strategies directly from the choices observed in the data (Aguirregabiria and Mira,

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2007, Bajari, Benkard and Levin, 2007, Pakes, Ostrovsky and Berry, 2007, Pesendorfer and Schmidt-Dengler, 2003).[1]

While our ability to estimate the dynamic model has significantly improved, in order to study counterfactual situations the equilibrium of the model still needs to be computed. Therefore, the original Ericson-Pakes model is somewhat limited in its application to cases where the state space is relatively small. In particular, the model is limited in its ability to study markets with multi-product firms. Below we show that in the setup of Pakes-McGuire (1994) with multi-product firms the state variable of each firm is a vector of qualities of each of its products. So even though the model might be tractable with single product firms, it quickly becomes non tractable with multi-product firms.[2] In this paper we propose an extension to the original model that allows the modeling of multi-product firms in dynamic games.

2 The model

We focus on the differentiated products version of the Ericson-Pakes model, detailed in Pakes-McGuire (1994).

2.1 Static Flow Profits

On the demand side, we assume that consumers choose one of the J products offered in the market, or the outside good, that gives the highest utility. The utility that consumer i obtains from purchasing brand j at time t is

$$U_{ijt} = \delta_{jt} + \epsilon_{ijt}, \qquad\qquad \delta_{jt} = x'_{jt}\beta - \alpha p_{jt} + \xi_{jt} \tag{1}$$

where δ_{jt} is the mean utility of product j at t; x_{jt} is a vector of observable characteristics of product j; p_{jt} is the price; the term ξ_{jt} captures product- and time-specific shocks which are

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[1] For related methods in the single agent context see Hotz and Miller (1993), Hotz, Miller, Sanders and Smith (1994), Manski (1993) and Aguirregabiria and Mira (2002). For a review of structural estimation of dynamic games see Ackerberg et al. (2006).

[2] Extending the core version to allow multi-product firms is also a concern listed in the agenda outlined by Pakes (2000, pg. 22).

common to all consumers; ϵ_{ijt} is an idiosyncratic error term, and α and β are parameters. If the consumer decides not to purchase any of the goods, she chooses the outside option that has a mean utility normalized to zero.

There are F firms in the market. Each firm f sells a subset of the J products denoted $\mathcal{F}_f$. We define the quality, or efficiency level, of a product as, $\omega_{jt} = x'_{jt}\beta + \xi_{jt}$, and the market structure of the industry at time t is characterized by a J-dimensional vector $s_t = (\omega_{1t}, ..., \omega_{Jt})$. The quantity sold and the optimal price will be a function of the efficiency levels of the firms' own products and the state of the industry (i.e., the competitors quality). Therefore, the static profit function of firm f can be written as (dropping subscripts t):

$$\pi_f\left(\{\omega_j\}_{j\in\mathcal{F}_f}, s\right) = \sum_{j\in\mathcal{F}_f}\left[p_j\left(\{\omega_j\}_{j\in\mathcal{F}_f}, s\right) - mc_j\right]\mathcal{M}\sigma_j\left(\{\omega_j\}_{j\in\mathcal{F}_f}, s\right) - C_f, \tag{2}$$

where p_j and σ_j, denote the price and market share of product j, mc_j and C_f are the marginal cost to produce product j and the fixed cost of production; $\mathcal{M}$ is the size of the market, including consumers who choose the outside option. We assume that firms set prices to maximize profits and the existence of a pure-strategy Bertrand-Nash equilibrium in prices.

2.2 Dynamic Decisions

In addition to pricing, in each period the firms decides if to invest and if so how much. Let x_j denote the investment in product j. Each unit of investment costs c, and the outcome is stochastic. Examples of investment are advertising or research that is aimed in improving the quality of the product. For clarity of exposition we do not consider entry or exit, either at the firm or at the product level.

The investment decisions are made to maximize the value of the firm, given by

$$V_f\left(\{\omega_{j,1}\}_{j\in\mathcal{F}_f}, s_1\right) = \max_{(x_{j,t}>0, j\in\mathcal{F}_f)}\sum_{t=1}^{\infty}\delta^{t-1}E\left[\pi_f\left(\{\omega_{j,t}\}_{j\in\mathcal{F}_f}, s_t\right) - c\sum_{j\in\mathcal{F}_f}x_{j,t}\right] \tag{3}$$

where δ is the discount rate.

The expectations are taken with respect to uncertainty about future quality levels, and competitors actions. Let the CDF $P\left(\left\{\omega_{j,t+1}\right\}_{j\in\mathcal{F}_f}, s_{t+1}\middle|\left\{x_{jt}\right\}_{j\in\mathcal{F}_f}, \left\{\omega_{jt}\right\}_{j\in\mathcal{F}_f}, s_t\right)$ represent firm f beliefs on next period efficiency levels $(\omega_{j,t+1})$ and market structure (s_{t+1}), given current investments (x_{jt}), efficiency levels (ω_{jt}), and market structure (s_t). In every period each product's efficiency evolves according to the following rule

$$\omega_{j,t+1} = \omega_{j,t} + (\nu_{j,t} - \zeta_t) \ , \tag{4}$$

where ν_{jt} and ζ_t are two independent, non-negative random variables. The first has a distribution that comes from a family $\{P(\cdot|x), x \in \mathcal{R}^+\}$ that is stochastically increasing in the investment level for that product, x_{jt}, and such that $\nu_{jt} = 0$ if $x_{jt} = 0$. The second is an exogenous random variable with probability $\mu(\zeta)$; in our setup it represents the efficiency value of the outside good, therefore it is a demand shock that is common to all products.

The value of the firm is a function of its own state and the state of its competitors. Even if each firm has a single product, there is a small number of firms, and the efficiency levels can take on a small number of values, then solving for the value function is subject to the curse of dimensionality. Pakes and McGuire (1994) propose to mitigate this problem somewhat by assuming exchangeability of the profit function such that the identity of the firms is not important. Therefore, only the number of firms at each efficiency level matters, not their identity. This significantly reduces the state space. However, with multi-product firms even if we are assume exchangeability, the state space will still be very large and probably not computationally tractable.

3 Results

In the section we propose an approach that makes the model tractable. The solution will rely heavily on what we will call an *adjusted inclusive value* (henceforth AIV) defined as

Definition 1 *Let* $i_f = log\left[\sum_{r\in\mathcal{F}_f} exp\left(\omega_r - \alpha mc_r\right)\right]$ *be the adjusted inclusive value (AIV) of firm f.*

The adjusted inclusive value is the difference between the quality of each product, defined by the characteristics, and the marginal cost needed to produce the quality level of each product. It can therefore be interpreted as the net quality level, or a value added of sort, that the firm is able to produce in the market. The AIV is closely related to the inclusive value (McFadden,1978), which captures the expected utility from several products prior to observing the random variables ϵ_{ij}'s, or equivalently the utility for the average consumer averaging over the ϵ_{ij}'s. From the firm perspective this inclusive value needs to be adjusted to take account of different marginal costs of production. Indeed, as we will now show under some assumptions the AIV is all that we need to compute the static profits.

Assumption A1 The idiosyncratic error term ϵ_{ijt} in (1) is identically and independently distributed type I extreme value.

Assumption A1 implies that aggregate demand is given by the Logit Model (McFadden, 1974). In particular it implies market shares of the form

$$\sigma_j\left(p;\{\omega_j\}_{j\in\mathcal{F}_f},s\right)=\frac{\exp(\omega_j-\alpha p_j)}{1+\sum_{k\in\mathcal{F}_f}\exp(\omega_k-\alpha p_k)}.$$

It is well-known that this model has several unattractive features (for example, see McFadden 1978; or Berry Levinsohn and Pakes, 1995). However, this assumption will turn out to be extremely useful for us. We discuss below ways to relax it. Note that this assumption is made by much of the literature cited in the Introduction.

We now show that under Assumption A1 the static flow profits can be written as a function of firm level AIV, and does not require the product-specific quality levels.

Lemma 1 *Under Assumption A1* $\pi_f\left(\{w_j\}_{j\in\mathcal{F}_f},s\right)=\pi_f\left(i_f,sf\right)$, *where* $sf=(i_1,...,i_F)$.

Proof: Taking the first-order condition of the profit function for firm f, as defined in (2), with respect to product j's price, we get

$$p-mc=\Omega^{-1}\sigma\left(p;\{w_j\}_{j\in\mathcal{F}_f},s\right)\tag{5}$$

where $\sigma(\cdot)$, p, and mc are $J \times 1$ vectors of market shares, prices, and marginal cost, respectively, and Ω is a $J \times J$ matrix with the element Ω_{jr} equal to $-\partial\sigma_r/\partial p_j$ if j and r are produced by the same firm, 0 otherwise. Given Assumption A1, the derivatives of the share equations are $\partial\sigma_j/\partial p_j = -\alpha\sigma_j(1-\sigma_j)$ and $\partial\sigma_r/\partial p_j = \alpha\sigma_j\sigma_r$. Plugging these back into equation (5) yields

$$(p - mc)_f = \frac{1}{\alpha\left(1 - \sum_{r\in\mathcal{F}_f}\sigma_r\right)} = \frac{1}{\alpha\left(1-\bar{\sigma}_f\right)}. \tag{6}$$

where $\bar{\sigma}_f = \sum_{r\in\mathcal{F}_f}\sigma_r$ is firm f's total share. This equation implies that each firm applies the same markup to all of its products. In order to compute the profits we need to compute the share of each firm.

We now show that, given this pricing rule, the share of firm f can be computed knowing only the firms' AIV.

$$\bar{\sigma}_f = \sum_{j\in\mathcal{F}_f}\sigma_j = \sum_{j\in\mathcal{F}_f}\frac{exp\left(\delta_j\right)}{1+\sum_{r=1}^{J} exp\left(\delta_r\right)} =$$

$$= \sum_{j\in\mathcal{F}_f}\frac{exp\left(-\alpha\left(p_j - mc_j\right)\right)exp\left(\omega_j - \alpha mc_j\right)}{1+\sum_{r=1}^{J} exp\left(-\alpha\left(p_r - mc_r\right)\right)exp\left(\omega_r - \alpha\, mc_r\right)}.$$

Since firms apply the same markup to each of their products,

$$= exp\left(-\alpha\, markup_f\right)\sum_{j\in\mathcal{F}_f}\frac{exp\left(\omega_j - \alpha\, mc_j\right)}{1+\sum_{g=1}^{F} exp\left(-\alpha\, markup_g\right)\sum_{r\in\mathcal{F}_g} exp\left(\omega_r - \alpha\, mc_r\right)},$$

$$= \frac{exp\left(i_f - \alpha\, markup_f\right)}{1+\sum_{g=1}^{F} exp\left(i_g - \alpha\, markup_g\right)}$$

where i_f is the AIV defined above. Firms' shares are function of the AIV, therefore, substituting the markup computed in equation (6) into the profit defined in equation (2) we get

$$\pi_f\left(\{w_j\}_{j\in\mathcal{F}_f}, s\right) = \mathcal{M}\frac{\bar{\sigma}_f\left(i_f, sf\right)}{\alpha(1-\bar{\sigma}_f\left(i_f, sf\right))} - C_f = \pi_f\left(i_f, sf\right) \qquad Q.E.D.$$

In order to show that the firm's dynamic problem also does not require the product-specific quality we need to make an additional assumption.

Assumption A2 $P\left(i_{f,t+1}, sf_{t+1} | \{x_{jt}\}_{j\in\mathcal{F}_f}, \{\omega_{jt}\}_{j\in\mathcal{F}_f}, s_t\right) = P\left(i_{f,t+1}, sf_{t+1} | x_{ft}, i_{ft}, sf_t\right)$, where $x_{f,t} = \sum_{j\in\mathcal{F}_f} x_{j,t}$.

This assumption restricts the stochastic evolution of the states in equation (4). There are several examples of the process described in equation (4) that will satisfy Assumption A2. For example, if the shocks $\nu_{jt} = \nu_{kt}$, or if $\nu_{jt}/\omega_{jt} = \nu_{kt}/\omega_{kt}$, for all k and j produced by the same firm. The assumption also restricts the way that investment decisions impact the transitions: all that matters is the total firm-level investment and not a the product specific decisions.

We now can write our main result.

Proposition 1 *Under Assumptions A1 and A2* $V_f\left(\{w_{j,1}\}_{j\in\mathcal{F}_f}, s_1\right) = V_f\left(i_f, sf_1\right)$, $\forall f \in \{1, ..., F\}$.

Proof. Substituting the result of Lemma 1 into equation (3) we get

$$V_f\left(\{\omega_{j,1}\}_{j\in\mathcal{F}_f}, s_1\right) =$$

$$= \max_{(x_{j,t}>0, j\in\mathcal{F}_f)} \sum_{t=1}^{\infty} \delta^{t-1} \int \left[\pi_f\left(i_{f,t}, sf_t\right) - c\sum_{j\in\mathcal{F}_f} x_{j,t}\right] dP\left(i_f, s_t | \{x_{j1}\}_{j\in\mathcal{F}_f}, \{\omega_{j1}\}_{j\in\mathcal{F}_f}, s_1\right)$$

By Assumption A2

$$= \max_{x_{f,t}>0} \sum_{t=1}^{\infty} \delta^{t-1} \int \left[\pi_f\left(i_{f,t}, sf_t\right) - c x_{f,t}\right] dP\left(i_{f,t}, s_t | x_{f1}, i_{f1}, sf_1\right) = V_f\left(i_f, sf_1\right) \qquad Q.E.D.$$

What we have shown is that, given our assumptions, the state variables of the problem include only firm level variables and do not require knowing, and keeping track of the product-level state variables. This result allows to consider firms that produce many brands without carrying the demand of each single brand, which would make the dynamic multi-product firm problem unfeasible.

4 Extensions

There are several ways to relax Assumptions A1 and A2 and still get some of the benefits of our approach. Assumptions A1 can be somewhat relaxed by assuming a generalized extreme value distribution. As a special case, consider the Nested Logit model. In this case we will need one state variable per firm per nest to compute the flow profits. Our approach will not work for the more general Random Coefficients Logit model, unless there is a limited number of combinations the characteristics can take. In this case we could compute the AIV for each unique combination, just like the nest in the Nested Logit Model. Obviously with a large number of different combinations this will not be very helpful. Assumption A2 can also be somewhat relaxed by allowing for other variables to enter the transition probabilities.

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